Elastic Turbulence in the Cahn-Hilliard-Navier-Stokes System

– A New Look at Familiar Themes in Turbulence

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- With:
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- Ackn:
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- See also Cahn, Ruiz and Nelson, Rahul Pandit and collaborators

What is It?

- System describing 2 <u>immiscible</u> fluids in 2D

$\psi = (\rho_A - \rho_B)/\rho_0$ → scalar field

2D CHNS

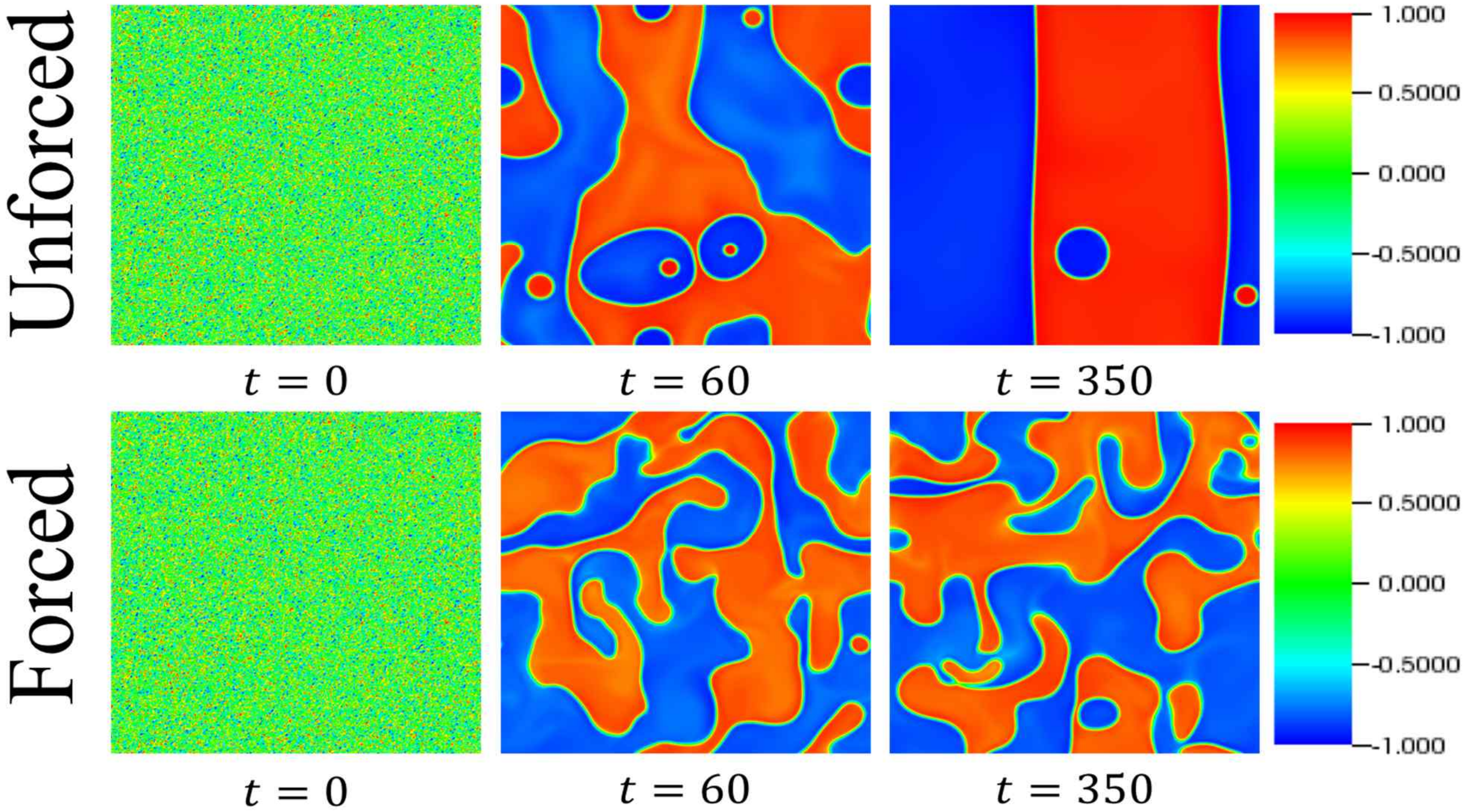
$$\begin{cases} \partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi) \\ \partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega \end{cases}$$

"Spinodal Decomposition"

→ Phase separation

What is It?

- Describes: Phase Separation



Why?

- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some "Fundamental" issues in plasma turbulence:
  - "Electromagnetics"
    - Most systems → 2D MHD, Reduced MHD + many linear effects
    - Physics of dual cascades and relaxation → relative importance, selective decay?
    - Physics of wave-eddy interaction effects on nonlinear transfer (Alfven effect ←→ Kraichnan)

Why?

- Zonal flow formation → negative viscosity phenomena → phase separation process
- "Blobby Turbulence" → how understand blob coalescence and relation to cascades

→ CHNS exhibits all of the above, with some new twists

Outline

- System and Applications
- Comparison/contrast to 2D-MHD
- Waves in CHNS
- Scales and Ranges
- Cascades and Spectra
- Key Issue: Real space vs k-space ←→ interfaces
- Single Eddy Dynamics: Mixing and “Flux Expulsion”
- Summary: Lessons learned

Scalar Field Equation

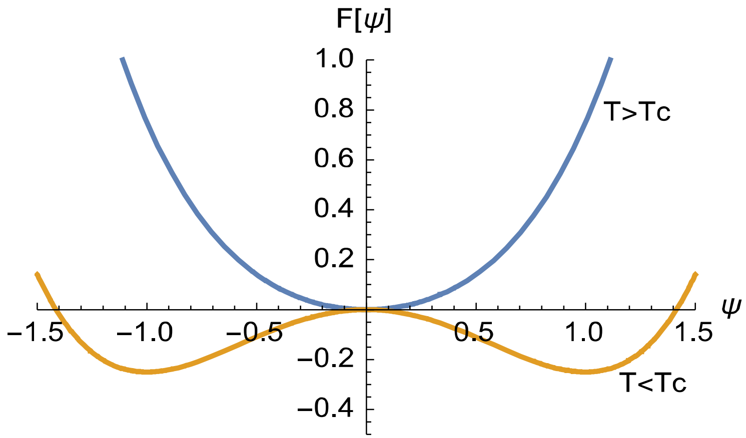
- Phase separation → second order phase transition -> Landau theory.
- Order parameter: local relative concentration:

$$\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$$

- $\psi = -1$ → A-rich phase, $\psi = 1$ → B-rich phase.
- Free energy functional:

$$\mathrm{F}(\psi) = \int d\vec{r} \left(\underbrace{\frac{1}{2}A\psi^2 + \frac{1}{4}B\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Curvature Penalty}} \right)$$

A, B, ξ parameters



Derivation, cont'd

- Isothermal $T < T_c$ without loss of generality, set B = −A = 1:

Free Energy

$$F(\psi) = \int d\vec{r}\left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\right)$$

- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$
- Fick's Law $\vec{J} = -D\nabla\mu$, continuity equation: $\frac{d\psi}{dt} + \nabla\cdot\vec{J} = 0$
- Combining:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

Cahn-Hilliard Equation

- Fluid velocity via convection term $d_t = \partial_t + \vec{v}\cdot\nabla$.

Derivation, cont'd

- Surface tension force enters via:

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} - \psi \nabla \mu + \nu \nabla^2 \vec{v}$$

- $\nabla \cdot \vec{v} = 0$ and 2D → CHNS equations

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

- $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$

- N.B. $\langle \vec{B} \rangle \leftrightarrow \langle \psi \rangle$ → mean density differential ←→ gradient in mean order parameter

2D CHNS vs. 2D MHD

- 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

$-\psi$: Negative diffusion term

ψ^3: Self nonlinear term

$-\xi^2 \nabla^2 \psi$: Hyper-diffusion term

With $\vec{v}=\hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$

$\psi \in [-1,1]$

- 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

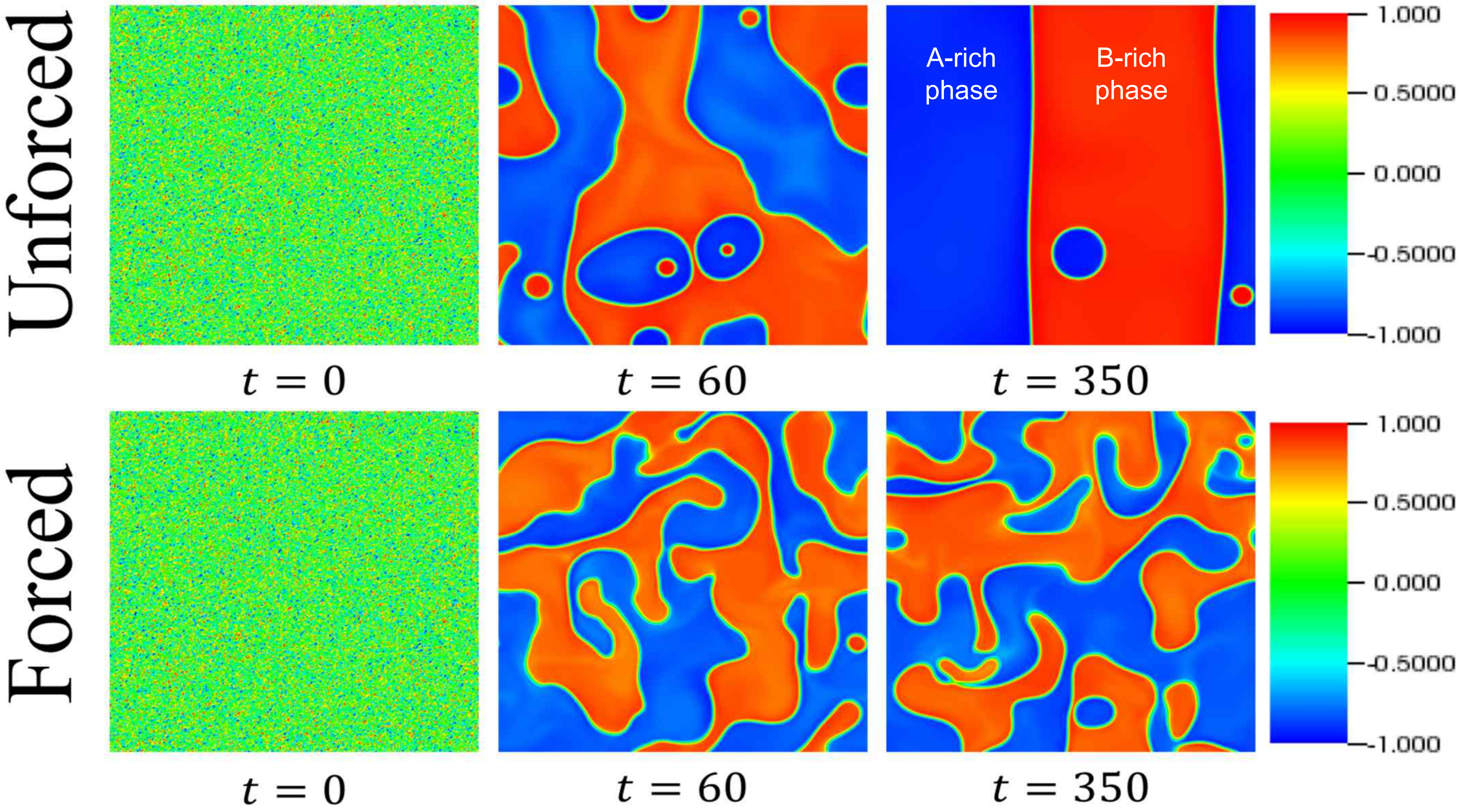
$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega$$

A: Simple diffusion term

With $\vec{v}=\hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{z} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$

Overview

→ Stirred Phase Separation ➔ "Blobby Turbulence"



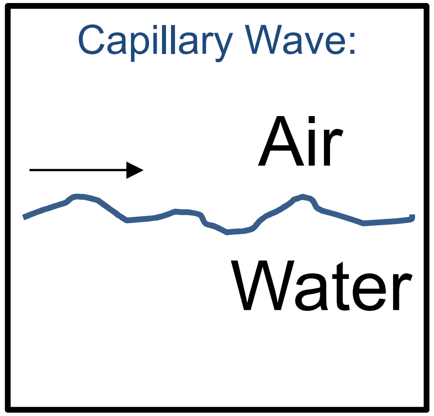
Waves in CHNS

- CNHS supports linear 'elastic' wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k} \times \vec{B}_{\psi 0}\right| - \frac{1}{2} i(CD + \nu)k^2$$

$$C \equiv [-1 - 6\psi_0 \nabla^2 \psi_0 / k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0 \nabla\psi_0 \cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2 k^2]$$

- Akin to capillary wave at interface:



- Propagates only along the interface between A, B. There $\nabla\rho \neq 0$ so

  $\vec{B}_0 = \nabla\langle\psi\rangle \times \hat{z} \neq 0$

- In contrast to MHD, elastic wave activity does not fill space

Energetics: Conserved Quantities

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

$$H^\psi = \int \psi^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

➔ Dual cascade expected!

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

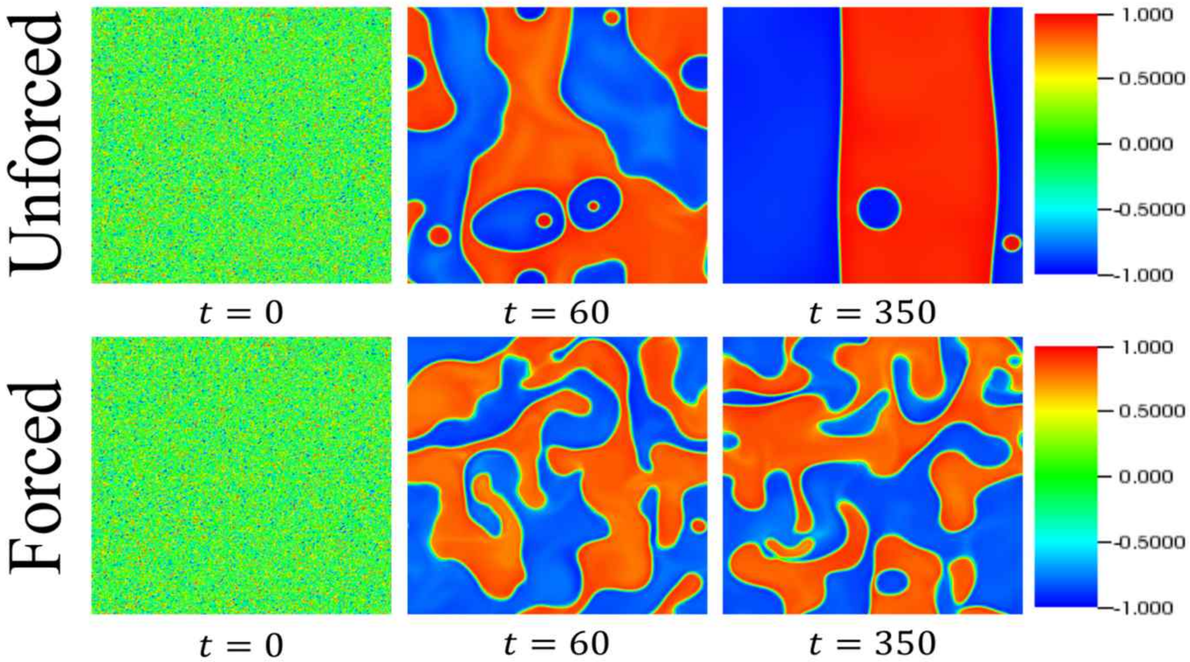
3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- “Ideal” here means $D, \eta = 0; \nu = 0$.

Scales, Ranges, Trends

Fluid straining
vs
Blob coalescence



- Fluid forcing → scale where turbulent straining ~ elastic restoring force (due surface tension)

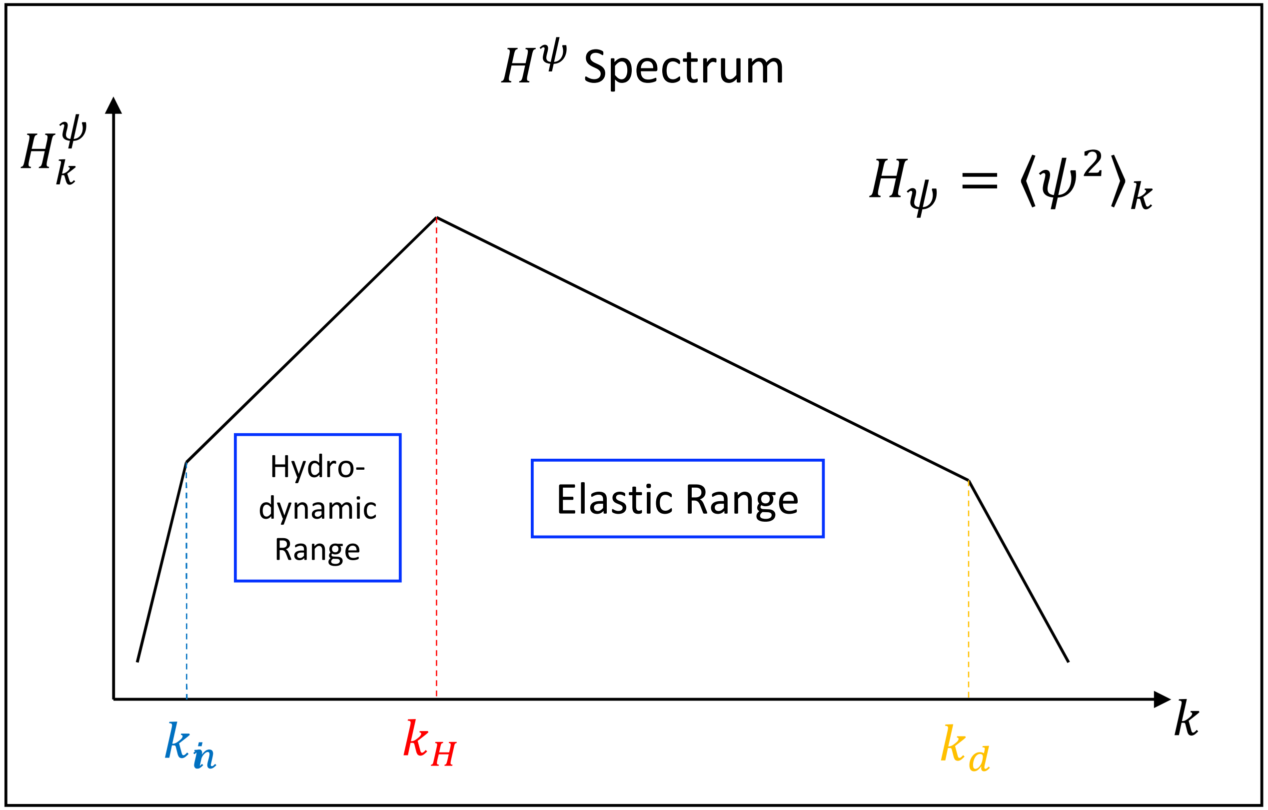
➔ **Hinze scale:** $L_H \sim \left(\frac{\rho}{\varepsilon}\right)^{-\frac{1}{3}} \epsilon_\Omega^{-2/9}$ } Scale where elastic effects enter

- $L_H < l < L_d$ ← dissipation

➔ elastic range

Scales, Ranges, Trends, cont'd

- $L_H/L_d \sim \left(\frac{\rho}{\varepsilon}\right)^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$ ➔ Extent of elastic range
- $L_H \gg L_d$ required for large elastic range → case of interest



- Key elastic range physics: Blob coalescence

Scales, Ranges, Trends, cont'd

- Blob coalescence tracked via mean scale evolution

$$L(t) \equiv 2\pi \left[\int dk\ S_k(k,t)\ /\ \int dk\ k\ S(k,t)\right]$$

$$S_k(k,t) \equiv \langle\, |\psi_k(k,t)|^2 \rangle \ \rightarrow\ \psi \text{ structure function}$$

- L grows via droplet coalescence

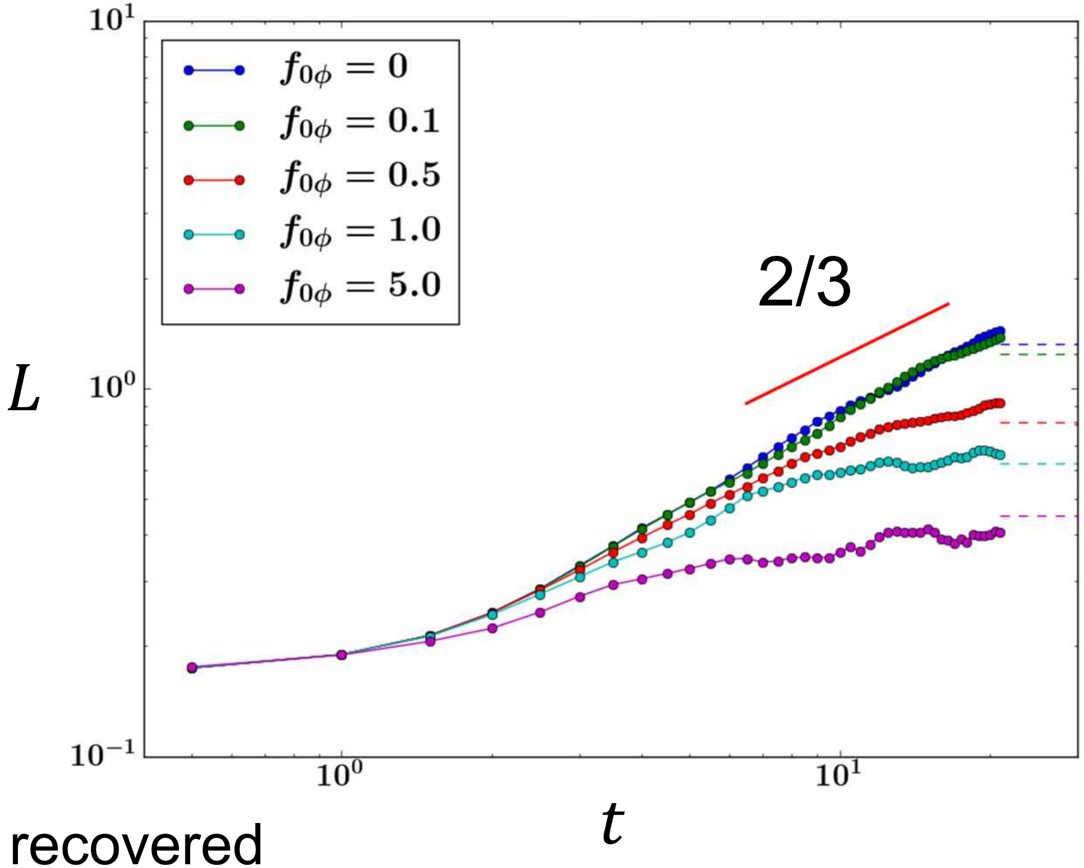
$$\vec{v} \cdot \nabla \vec{v} \sim \frac{\varepsilon^2}{\rho}\, \nabla^2 \psi \nabla \psi \Rightarrow \frac{\dot{L}^2}{L} \sim \frac{\sigma}{\rho} \left(\frac{1}{L^2} \right)$$

$$L(t) \sim t^{\frac{2}{3}}$$

- With external forcing, blob coalescence arrested at Hinze scale

Scales, Ranges, Trends, cont'd

- Heuristic blob size evolution scaling confirmed:

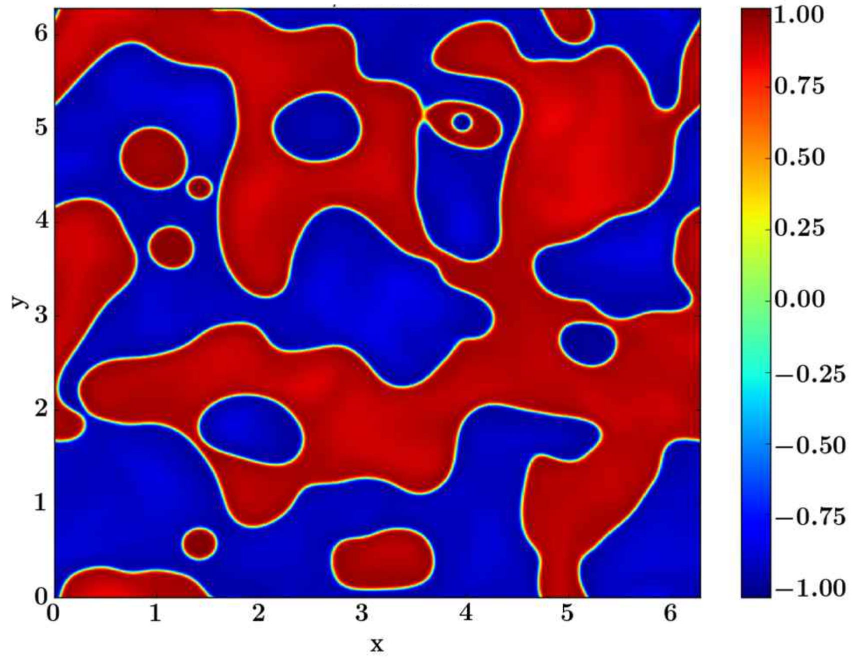


Hinze scale values for different forcing

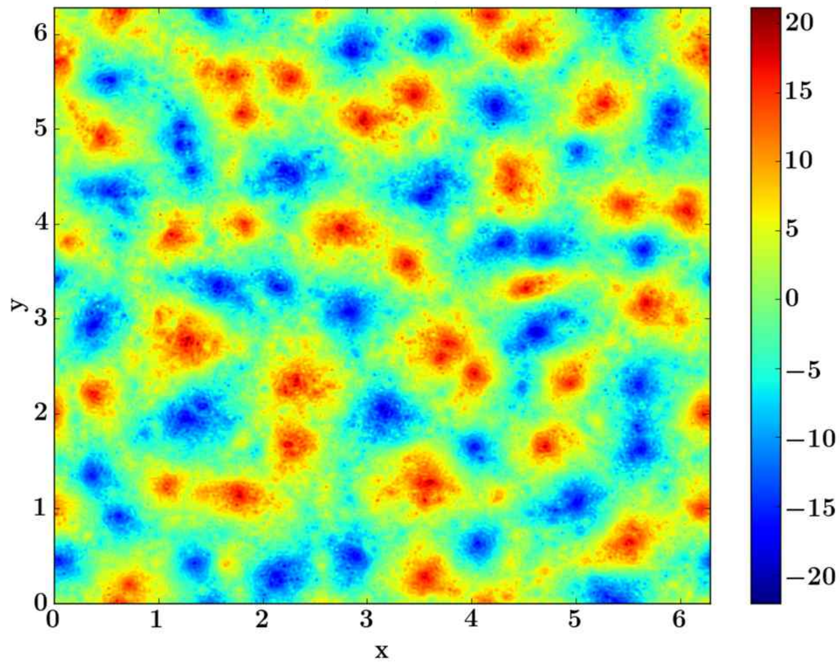
  - $L(t) \sim t^{2/3}$ recovered
  - Blob growth arrest observed
  - Blob growth saturation scale tracks Hinze scale

Scales, Ranges, Trends, cont'd

ψ CHNS



A MHD



- Blob coalescence in CHNS analogous to flux coalescence in MHD
- Suggests <u>inverse cascade</u> of $H^{\psi} = \langle \psi^2 \rangle$ in CHNS
- Supported by equilibrium statistical mechanics studies $[k_{min} < |k| < k_{max}]$

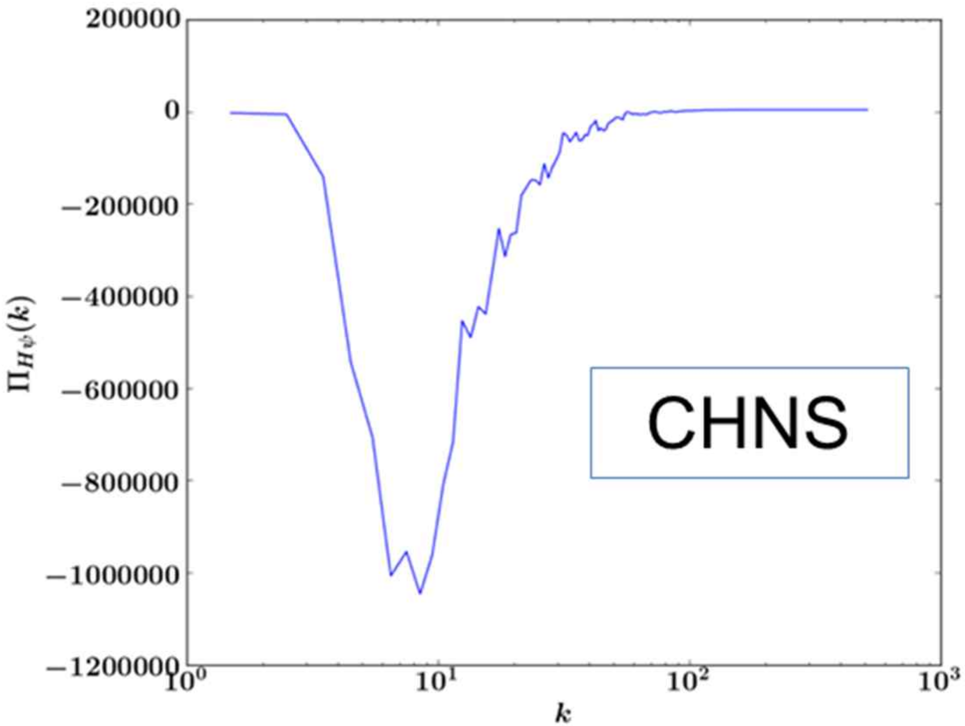
  Multiple IOMs

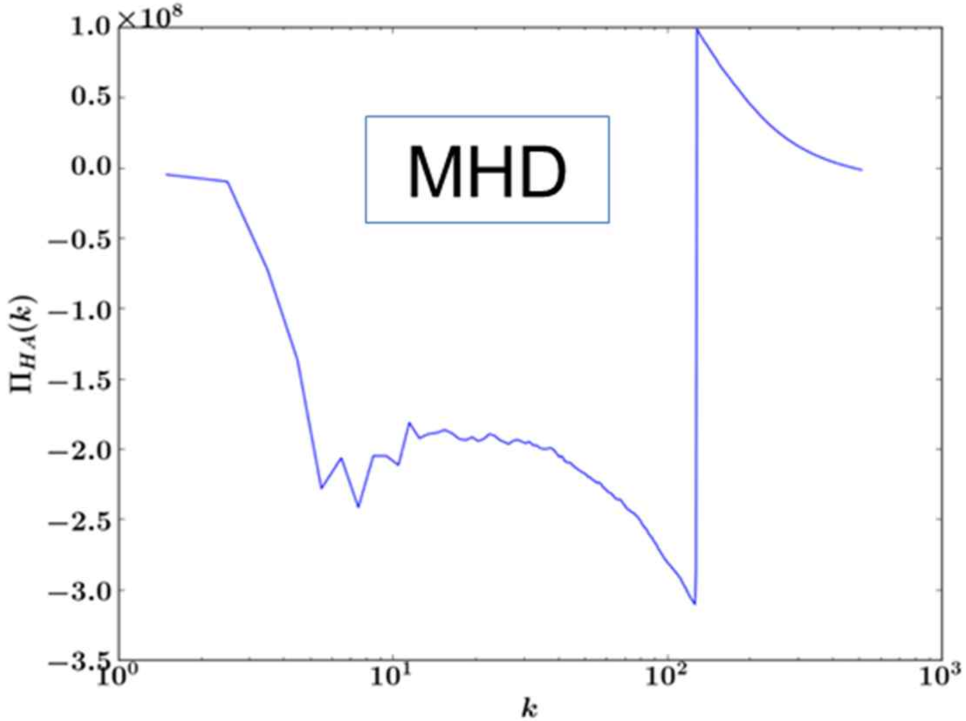
Cascades

- Dual cascade:
  - Inverse cascade of $\langle\psi^2\rangle_k$
  - Forward cascade of E_k
- Inverse cascade of $\langle\psi^2\rangle$ is formal expression of blob coalescence process ➔ generate larger scale structures till limited by straining
- Forward cascade of E as usual, as elastic force breaks enstrophy conservation

Cascade, cont'd

- Spectral flux of $\langle \psi^2 \rangle_k$



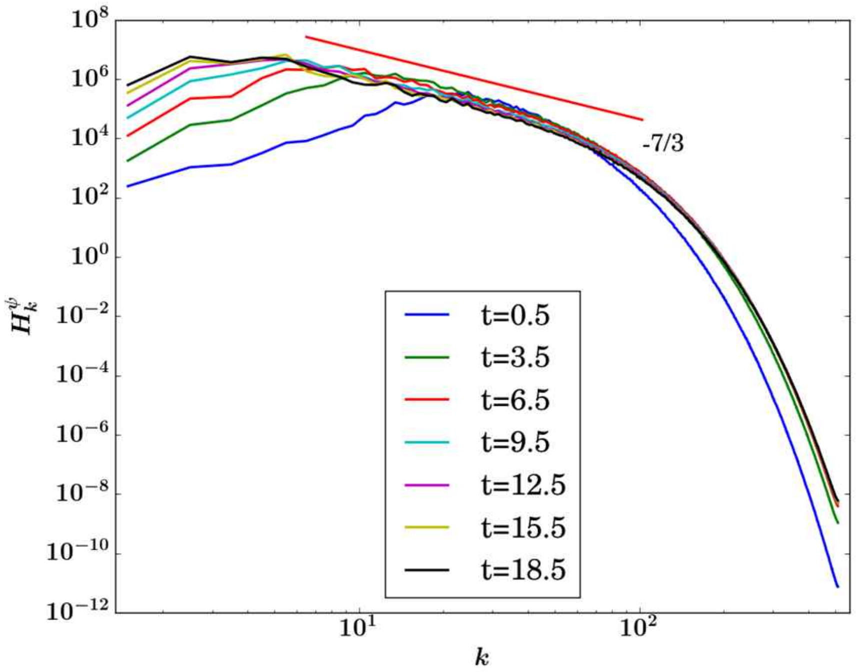


- CHNS: ψ is unforced → natural aggregation
- MHD: weak small scale forcing on A drives inverse cascade
- Both fluxes negative ➔ inverse cascade; H^{ψ}, H^{A}

Cascade, cont'd

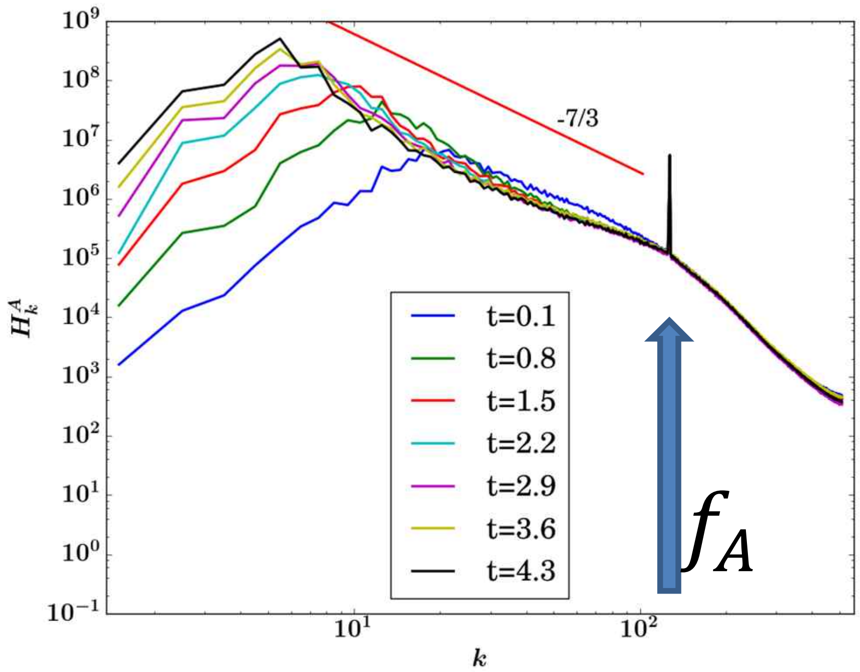
- Inverse cascade spectra

CHNS



$$H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}$$

MHD

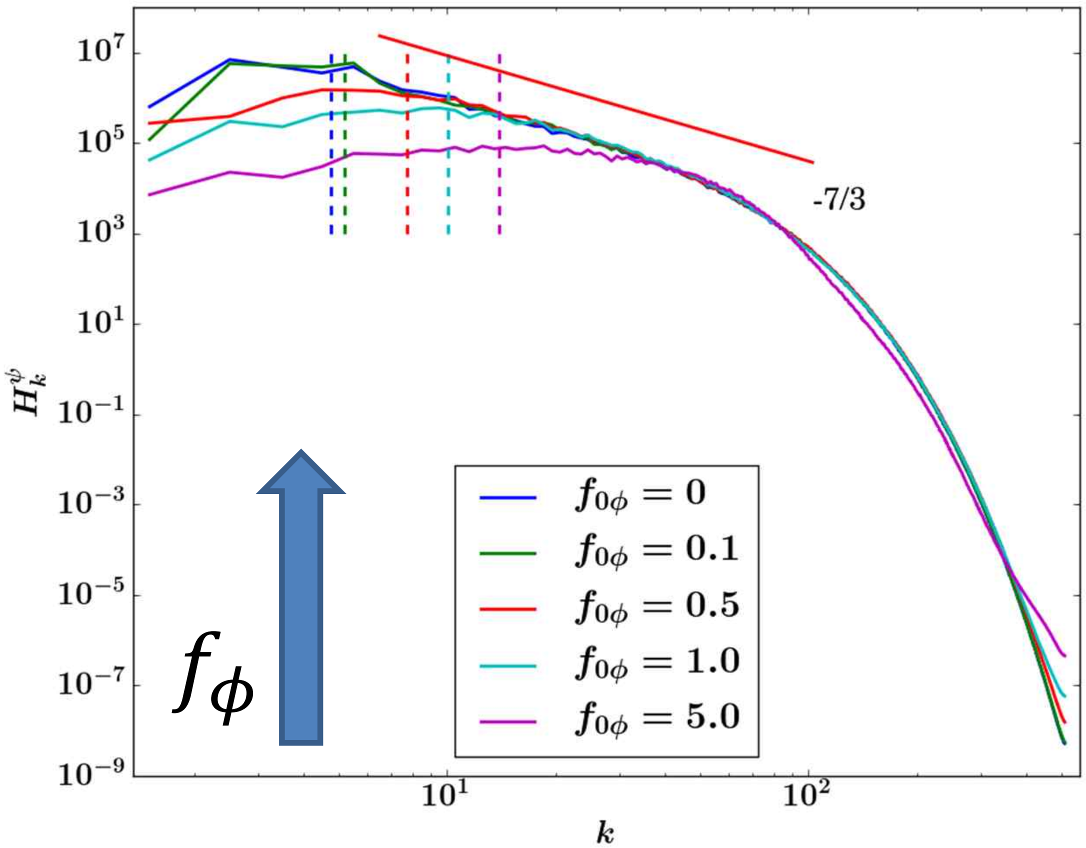


$$H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}$$

- MHD is weakly forced in A, at small scale
- Both systems exhibit $k^{-7/3}$ spectra

Cascade, cont'd

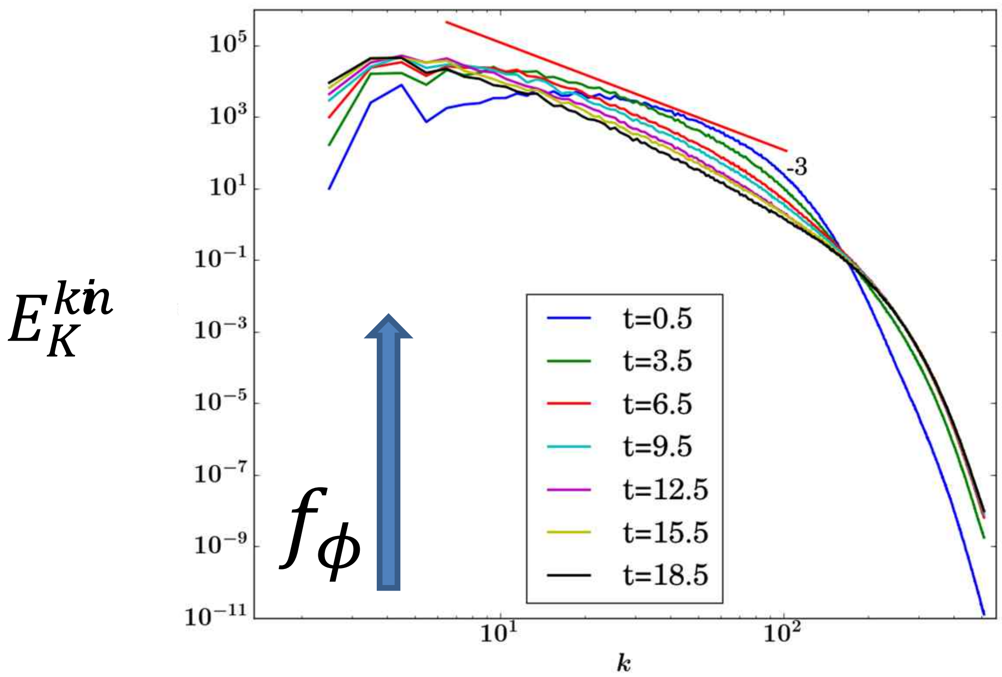
- Inverse cascade of $\langle\psi^2\rangle$ exhibits same power law scaling, so long as $L_H \gg L_d$, maintaining elastic range: Robust process



- Obtain both spectra via constant transfer, assuming Elastic/Alfvenic energy 'balance'

Cascade, cont'd

- Energy spectrum



- $E^k \sim k^{-3}$
  - Closer to enstrophy cascade range scaling, in 2D Hydro
  - Marked departure from expected $k^{-3/2}$ for MHD. Why?

Crux of the Matter

- Why does CHNS ←→ MHD correspondence hold well for $H_\psi \sim H_A \sim k^{-7/3}$ yet break down drastically for energy?
- What physics underpins this surprise?

➔

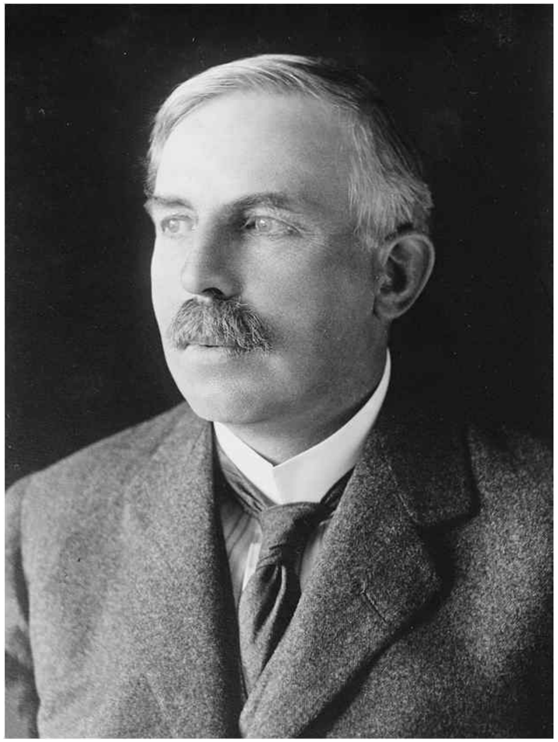
- Need understand differences, as well as similarities, between CHNS and MHD problems.

analogies

"We have run out of ~~money~~.

Its time to start thinking".

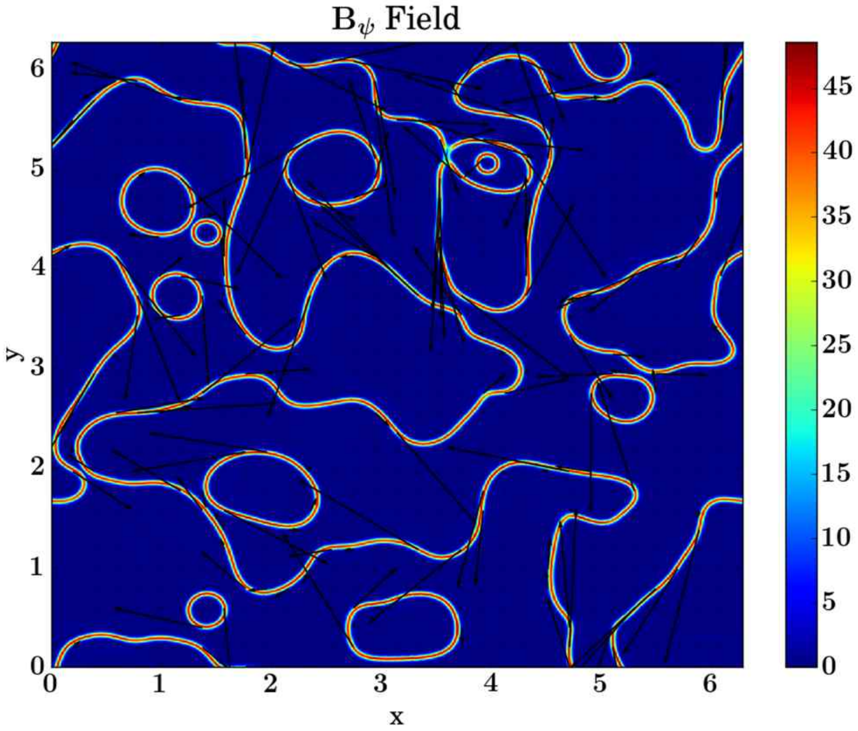
- after E.O. Rutherford



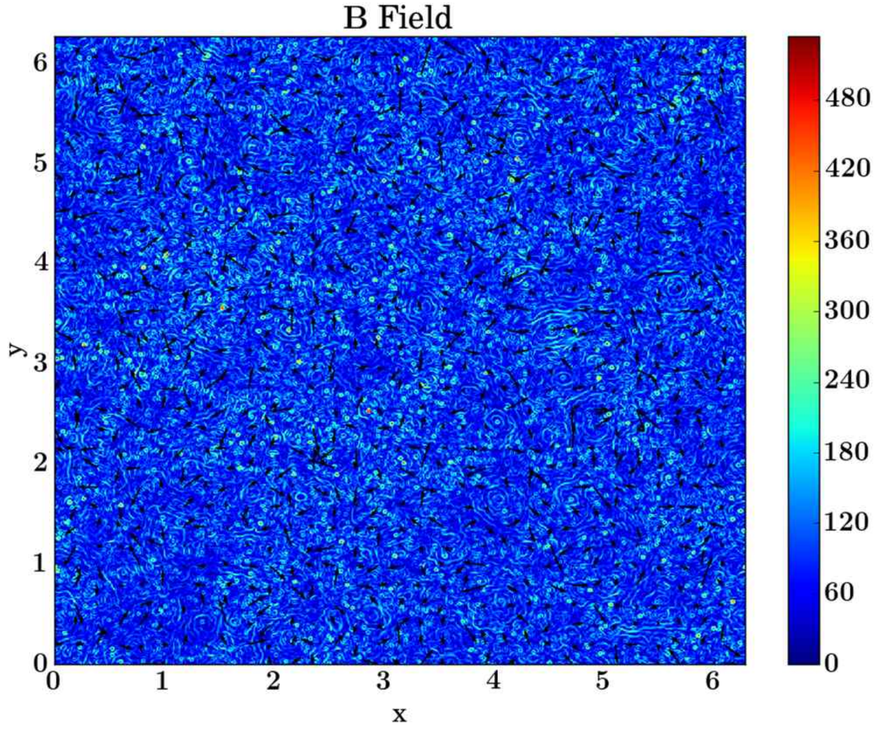
Resolution

- Elastic back-reaction is limited to regions of density contrast i.e. $\nabla\psi \sim B_\psi \neq 0$
- As blobs coalesce, interfacial region diminished. ‘Active region’ of elasticity decays
- In MHD, fields pervade system

CHNS



MHD



Resolution, cont'd

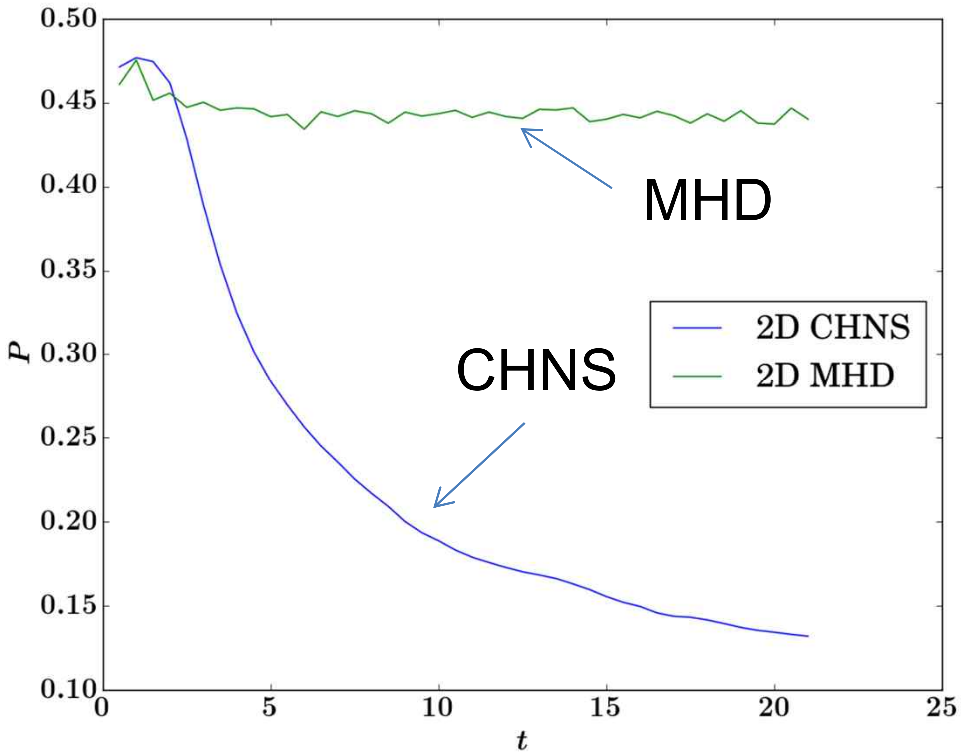
- Define interface packing fraction P

$$P = \frac{\text{\# locations where } |B_\varphi| > B_\varphi^{rm\,s}}{\text{Total \# locations}}$$

"locations" ≡ mesh grid cells

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

P_{CHNS} decays in time while P_{MHD} stationary



Elastic forces weaken

What is the Lesson?

- Avoid power law tunnel vision!
- Real space realization of the flow necessary to understand key dynamics. Track interfaces and packing function P.
- One player in dual cascade (i.e. $\langle\psi^2\rangle$) can modify or constrain the dynamics of the other (i.e. E).
- Against conventional wisdom, $\langle\psi^2\rangle$ inverse cascade due blob coalescence is the robust nonlinear transfer process in CHNS turbulence.

Broader Implications ←→ Speculations

- What, really, is the essential transfer process in MHD?

  i.e. theoretical focus is overwhelmingly on *Energy*

  – Follows fluids, examine energy with forcing in $\vec{v}$ equation

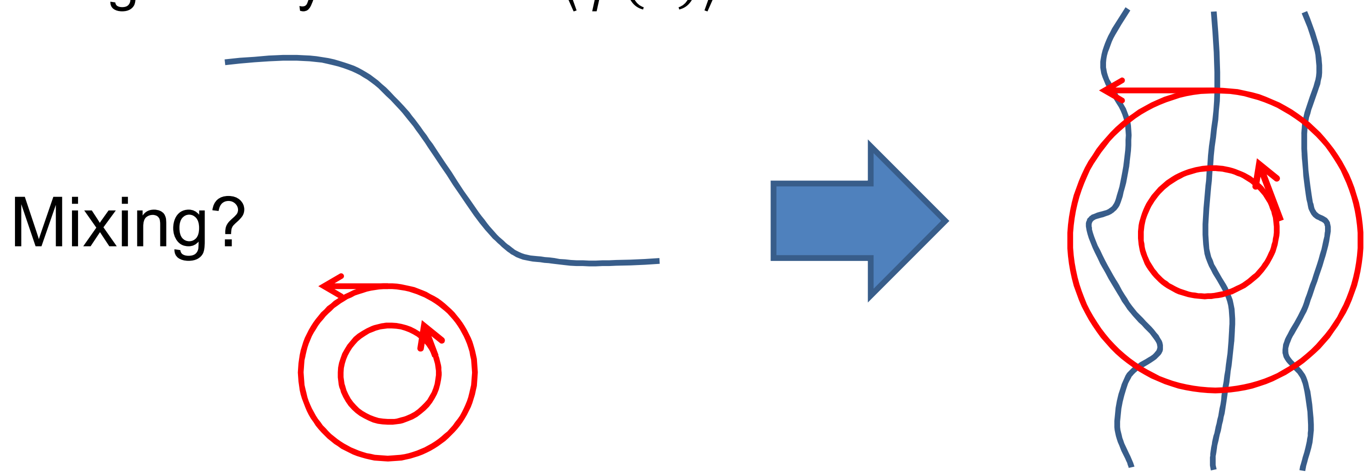
  but

  – Alfven's theorem is key constraint in MHD. So, is inverse cascade $\langle A^2 \rangle$ (or $\langle \vec{A} \cdot \vec{B} \rangle$) really fundamental?

- Can dual cascade process interact?
- Can 2D MHD turbulence be thought of as flux aggregation vs. fragmentation competition?

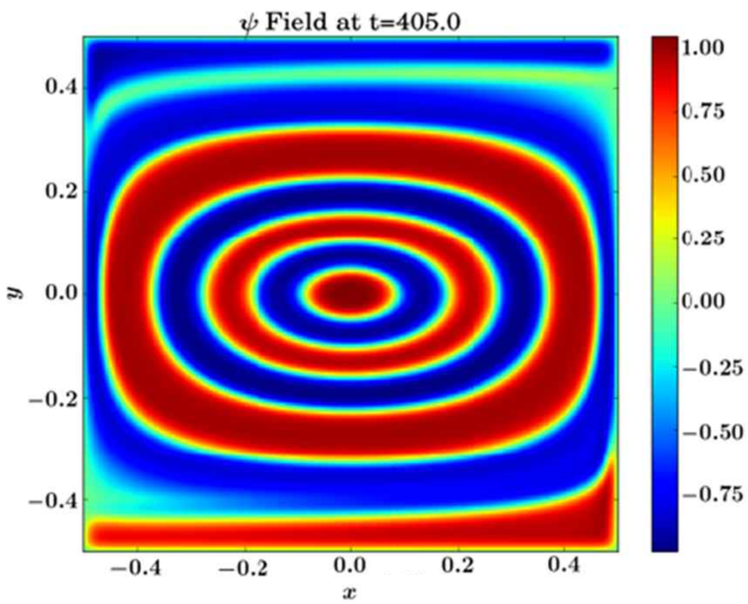
Related Work

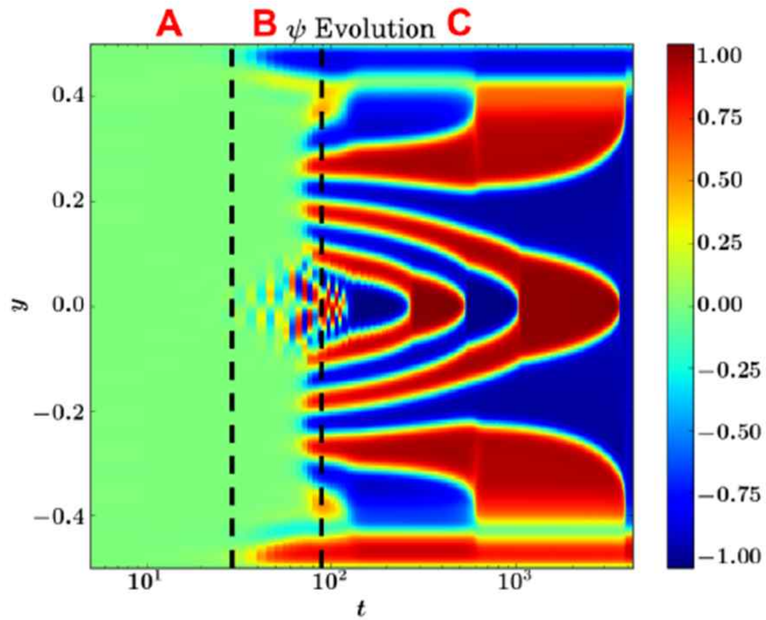
- Single eddy in linear $\langle\psi(x)\rangle$



Analogous to flux expulsion in MHD (Weiss, '66)

- ψ homogenized, but metastable target patterns formed and merge. $\tau \sim (\varepsilon^2 D)^{-1/5}$





Conclusion

- Turbulent spinodal decomposition dynamics illuminates familiar themes in physics of MHD cascades, relaxation, and selective decay, from a novel perspective
- Blob coalescence is dominant process in CHNS
- Real space configuration and packing of interfaces are essential to physics of dual cascade

See: Fan, P.D. et. al. Phys. Rev. Fluids 2016
Phys. Rev. E, 2017

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